Q1. *[Warm-up.]* Write down all the permutations in the symmetric group S_3. Express your answer in *matrix form*. For example, the permutation of $\{1, 2, 3\}$ given by $1 \to 2$, $2 \to 3$ and $3 \to 1$ is represented in matrix form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Solution:

There are $3! = 6$ permutations in S_3. In matrix form, the are:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Q2. *[Multiplication of permutations.]* In S_3, let

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Determine the following permutations and express your answer in matrix form.

(a) se.

(b) es.

(c) s^2. (This is ss.)

(d) st.

(e) ts.

Solution:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$.

Notice that $st \neq ts$.

Q3. *[Cycle notation.]* The following are permutations in S_4 that have been expressed in matrix form. Express them in cycle notation.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}$.

Solution:

(a) (1)(2)(3)(4).

(b) (12)(34).

(c) (1234).

(d) (124)(3).

Q4. *[Inverses.]* Referring back to Q2, determine the following permutations. Express your answers in both matrix form **and** cycle notation.

(a) e^{-1}.

(b) s^{-1}.

(c) t^{-1}.

(d) $(st)^{-1}$. [*Question:* How, if at all, is $(st)^{-1}$ related to s^{-1} and t^{-1}?]

Solution:

Let's first express e, s, t in cycle notation. We have:

$$e = (1)(2)(3), \quad s = (123) \quad \text{and} \quad t = (1)(23).$$

(a) $e^{-1} = e$. In matrix form, this is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$. In cycle notation, this is $(1)(2)(3)$.

(b) $s^{-1} = (321)$. In matrix form, this is $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

(c) $t^{-1} = t$. In matrix form, this is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$. In cycle notation, it's $(1)(23)$.

(d) We've computed that $st = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$. In cycle notation, this is $(12)(3)$. So its inverse is itself. That is, $(st)^{-1} = (12)(3)$.
We also notice that $(st)^{-1} = t^{-1}s^{-1}$. Indeed,

$$t^{-1} = (1)(23) \quad \text{and} \quad s^{-1} = (321).$$

So

$$t^{-1}s^{-1} = (1)(23)(321) = (12)(3).$$

The way to think about the fact that $(st)^{-1} = t^{-1}s^{-1}$ is as follows. If you want to undo the effect of st (which—remember—is: *apply t and then apply s*) then you should first *unapply s* then *unapply t*! If you put on your socks then shoes, you first take off your shoes then you take off your socks. You don't take off your socks before your shoes! (Right..?)

Q5. *[Commutators]* An important concept in the proof of the insolvability of the quintic is the notion of *commutators* of permutations. A commutator of two permutations $s, t \in S_n$ is a permutation of the form

$$t^{-1}s^{-1}ts.$$

Reading from right to left, we interpet this permutation as

perform s, then perform t, then unperform s, then unperform t.

Since the order of multiplication in S_n is important, the above is **not** the same as $s^{-1}t^{-1}ts$, which would result in the identity permutation.

(a) Let $s = (12)$ and $t = (13)$ in S_3. Find the commutator $t^{-1}s^{-1}ts$.

(b) **[Challenging!]** Show that $(1\ 2\ 3\ 4\ 5)$ is a commutator of permutations in S_5. That is, find permutations s and t in S_5 such that

$$(1\ 2\ 3\ 4\ 5) = t^{-1}s^{-1}ts.$$

[**Hint:** There are several possible t and s. Try to find s given that $t = (2\ 3\ 5\ 1\ 4)$.]

Note: A key step in the proof of the insolvability of the quintic is to show that *every* permutation in S_5 is a commutator of two other permutations. (This is not true in S_2, S_3 and S_4, and is part of the reason why we have quadratic, cubic and quartic formulas!)

Solution:

(a) The inverses of s and t are themselves: $s^{-1} = s$ and $t^{-1} = t$. So,

$$t^{-1}s^{-1}ts = (13)(12)(13)(12) = (132).$$

(b) Starting with $t == (2\ 3\ 5\ 1\ 4)$, after some trial and error, we arrive at $s = (3\ 1\ 5\ 2\ 4)$. Indeed,

$$t^{-1}s^{-1}ts = (41532)(42513)(23514)(31524) = (34512).$$